

# Left-orderings on free products of groups

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## Abstract

We show that no left-ordering on a free product of (left-orderable) groups is isolated. In particular, we show that the space of left-orderings of a free product of finitely generated groups is homeomorphic to the Cantor set. With the same techniques, we also give a new and constructive proof of the fact that the natural conjugation action of the free group (on two or more generators) on its space of left-orderings has a dense orbit.

## Introduction

A (non-necessarily total) order relation  $\preceq$  on a group  $\Gamma$  is said to be a *partial-left-ordering*, if for every  $\gamma_1, \gamma_2, \gamma_3$  in  $\Gamma$ , we have that  $\gamma_1 \prec \gamma_2$  implies  $\gamma_3\gamma_1 \prec \gamma_3\gamma_2$ . An element  $\gamma \in \Gamma$  is called  $\preceq$ -positive (resp.  $\preceq$ -negative) if  $id \prec \gamma$  (resp.  $\gamma \prec id$ ). The subset of  $\preceq$ -positive elements, usually called the *positive cone* for  $\preceq$ , will be denoted by  $P_{\preceq}$ . Clearly,  $P_{\preceq}$  satisfies

- (O1)  $P_{\preceq}P_{\preceq} \subseteq P_{\preceq}$ , that is,  $P_{\preceq}$  is a semi-group, and
- (O2)  $P_{\preceq} \cap P_{\preceq}^{-1} = \emptyset$ , where  $P_{\preceq}^{-1} = \{g^{-1} \in \Gamma \mid g \in P_{\preceq}\} = \{g \in \Gamma \mid g \prec id\}$ .

If in addition,  $\preceq$  is a total order, we will simply say that  $\preceq$  is a *left-ordering*. In this case, the set of  $\preceq$ -positive elements also satisfies

- (O3)  $\Gamma = P_{\preceq} \cup P_{\preceq}^{-1} \cup \{id\}$ .

Conversely, given any subset  $P \subseteq \Gamma$  satisfying the conditions (O1), (O2) and (O3) (resp. (O1) and (O2)) above, we can define a left-ordering (resp. a partial-left-ordering)  $\preceq_P$  by letting  $f \prec_P g$  if and only if  $f^{-1}g \in P$ . We will usually identify  $\preceq$  with  $P_{\preceq}$ .

Given a group  $\Gamma$  (of arbitrary cardinality), we denote the set of all partial-left-orderings on  $\Gamma$  by  $\mathcal{PLO}(\Gamma)$ . This set has a natural topology first exploited by Sikora for the case of (total orderings on) countable groups [20]. This topology can be defined by identifying  $P \in \mathcal{PLO}(\Gamma)$  with its characteristic function  $\chi_P \in \{0, 1\}^\Gamma$ . In this way, we can view  $\mathcal{PLO}(\Gamma)$  embedded in  $\{0, 1\}^\Gamma$ . This latter space, with the product topology, is a Hausdorff, totally disconnected, and compact space. It is not hard to see that (the image of)  $\mathcal{PLO}(\Gamma)$  is closed inside, and hence compact as well (see [11, 13, 15, 20] for details). In the same way, for a left-orderable group  $\Gamma$ , the space of all left-orderings, here denoted  $\mathcal{LO}(\Gamma)$ , is closed inside  $\mathcal{PLO}(\Gamma)$ , hence compact as well. In [11], it is shown that  $\mathcal{LO}(\Gamma)$  is either finite or uncountable.

A basis of neighborhoods of  $\preceq$  in  $\mathcal{LO}(\Gamma)$  is the family of the sets  $V_{f_1, \dots, f_k} = \{\preceq' \in \mathcal{LO}(\Gamma) \mid id \prec' f_i, \text{ for } i = 1, \dots, k\}$ , where  $\{f_1, \dots, f_k\}$  runs over all finite subsets of  $\preceq$ -positive elements of  $\Gamma$  (the same being true for  $\mathcal{PLO}(\Gamma)$ ). Therefore, it is natural to say that a left-ordering  $\preceq$  of  $\Gamma$  is *isolated* if and only if there is a finite family  $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$  such that  $\preceq$  is the only left-ordering of  $\Gamma$  with the property that  $\gamma_i \succ id$ , for  $1 \leq i \leq n$ .

Knowing whether a given group has an isolated left-ordering turns out to be a natural and old question in the theory of left-orderable groups (although not always expressed in topological terms...). A major progress in the understanding of groups having isolated left-orderings, is the

classification of groups admitting only finitely many left-orderings (all of them isolated) made by Tararin [10, Theorem 5.2.1]. In addition, we count with the remarkable examples of groups admitting infinitely many left-orderings together with some isolated left-orderings, such as the braid groups [5] (see however [16]), and the groups appearing in [8, 14]. On the other hand, it is known that some classes of groups, such as nilpotent groups [15] (more generally, left-orderable groups of sub-exponential growth [19, Remark 2.2.3]) and the groups appearing in [18], have no isolated left-orderings unless they have only finitely many left-orderings.

In the case of the free group of finite rank  $F_n$ ,  $n \geq 2$ , it was proved by McCleary [12] that  $F_n$  has no isolated left-orderings<sup>1</sup>. McCleary's proof relies on the study of the so called free-lattice-ordered group (in his case) over the free group, which is a universal object introduced by Conrad in [2]. An independent proof of this fact was given by Navas in [15], where he studies the so-called *dynamical realization of a left-ordering* (see §1) of  $F_n$ , which is an order-preserving action on the real line that encodes all the information of the given left-ordering.

In this article, we simplify and generalize Navas' approach to get a generalization of McCleary's result for the case of free products of left-orderable groups. (Recall that the free product of left-orderable groups is left-orderable [10, Corollary 6.1.3].) We show

**Theorem A:** *Let  $G$  and  $H$  be two left-orderable groups. Then the free product  $G * H$  has no isolated left-orderings.*

To prove Theorem A we first work the case where  $G$  and  $H$  are finitely generated §2.1. Then, in §2.2, we use the compactness of  $\mathcal{PLO}(\Gamma)$  to provide an argument ensuring Theorem A. We note that Theorem A does not extend to the case of amalgamated free products, since the groups with isolated left-orderings appearing in [8, 14] (for instance, the braid group  $B_3$ ) are of that form.

A direct consequence of Theorem A is that no positive cone of a left-ordering on a free product of groups is finitely generated as a semigroup (see for instance [15, Proposition 1.8]). However, the converse to this is not true. In §2.3, we show that  $\langle a, b \mid bab^{-1} = a^{-2} \rangle$  is a group with an isolated left-ordering whose positive cone is not finitely generated as a semigroup.

Besides its compactness,  $\mathcal{LO}(\Gamma)$  has another very important property, namely, that the group  $\Gamma$  naturally acts on it by conjugation:

$$\gamma(\preceq) = \preceq_\gamma, \text{ where } \gamma_1 \prec_\gamma \gamma_2 \text{ if and only if } \gamma\gamma_1\gamma^{-1} \prec \gamma\gamma_2\gamma^{-1}.$$

This action turns out to be by homeomorphisms since  $\gamma(V_{\gamma_1, \dots, \gamma_k}) = V_{\gamma\gamma_1\gamma^{-1}, \dots, \gamma\gamma_k\gamma^{-1}}$ . This action was defined by Ghys and was first exploited in [13] by Morris-Witte.

In [1], Clay found a strong connection between the conjugation action of  $\Gamma$  on its space of left-orderings and some natural representations of the free-lattice-ordered group over  $\Gamma$ . In the special case of a free group, this connection, together with a previous result of Kopytov [9], allowed him to show

**Theorem B (Clay):** *Let  $\mathcal{F}$  be a free group of countable rank greater than one. Then, the space of left-orderings of  $\mathcal{F}$  has a dense orbit under the natural conjugation action of  $\mathcal{F}$ .*

Nevertheless, his proof is highly non-constructive, and Kopytov's result also involves the free-lattice-ordered group over the free group. In Section 3 of this work, we use our dynamical machinery to give an explicit and self-contained construction of a left-ordering on  $\mathcal{F}$  whose set of conjugates is dense. However, our method does not solve the following question, that may have some interest in rigidity theory.

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<sup>1</sup>The fact that the free groups of infinite rank has no isolated left-orderings is easy and appears, for instance, in [4].

**Question:** Does  $\mathcal{F}$  admits a dense orbit for the diagonal action on  $\mathcal{LO}(\mathcal{F}) \times \mathcal{LO}(\mathcal{F})$ ?

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## 1 The dynamical realization of a left-ordering

Though orderability may look as a very algebraic concept, it has a deep (one-dimensional) dynamical content. For instance, a group is left-orderable if and only if it embeds in the group of order-preserving automorphisms of a totally ordered set  $\Omega$ ; see for instance [10, Theorem 3.4.1].

For the case of countable groups (*e.g.* finitely generated), we can give more dynamical information since we can take  $\Omega$  as being the real line (see [6, Theorem 6.8], or [15] for further details).

**Proposition 1.1.** *For a countable infinite group  $\Gamma$ , the following two properties are equivalent:*

- $\Gamma$  is left-orderable,
- $\Gamma$  acts faithfully on the real line by orientation-preserving homeomorphisms. That is, there is an homomorphic embedding  $\Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$ .

*Sketch of proof:* To show that a subgroup of  $\text{Homeo}_+(\mathbb{R})$  is left-orderable, we construct what is usually called an *induced left-ordering*. To do this, we take a dense sequence  $(x_0, x_1, \dots)$  of points in  $\mathbb{R}$ , and we define  $\preceq_{(x_0, x_1, \dots)}$  by declaring

$$\gamma \succ_{(x_0, x_1, \dots)} id \quad \text{if and only if} \quad \gamma(x_i) > x_i,$$

where  $i = \min\{j \mid x_j \neq \gamma(x_j)\}$ . Showing that  $\preceq_{(x_0, x_1, \dots)}$  is a total left-ordering is routine.

For the converse, we construct what is called a *dynamical realization of a left-ordering*  $\preceq$ . Fix an enumeration  $(\gamma_i)_{i \geq 0}$  of  $\Gamma$  such that  $\gamma_0 = id$ , and let  $t_{\preceq}(\gamma_0) = 0$ . We shall define an order-preserving map  $t_{\preceq} : \Gamma \rightarrow \mathbb{R}$  by induction. Suppose that  $t_{\preceq}(\gamma_0), t_{\preceq}(\gamma_1), \dots, t_{\preceq}(\gamma_i)$  have been already defined. Then if  $\gamma_{i+1}$  is greater (resp. smaller) than all  $\gamma_0, \dots, \gamma_i$ , we define  $t_{\preceq}(\gamma_{i+1}) = \max\{t_{\preceq}(\gamma_0), \dots, t_{\preceq}(\gamma_i)\} + 1$  (resp.  $\min\{t_{\preceq}(\gamma_0), \dots, t_{\preceq}(\gamma_i)\} - 1$ ). If  $\gamma_{i+1}$  is neither greater nor smaller than all  $\gamma_0, \dots, \gamma_i$ , then there are  $\gamma_n, \gamma_m \in \{\gamma_0, \dots, \gamma_i\}$  such that  $\gamma_n \prec \gamma_{i+1} \prec \gamma_m$  and no  $\gamma_j$  is between  $\gamma_n, \gamma_m$  for  $0 \leq j \leq i$ . Then we set  $t_{\preceq}(\gamma_{i+1}) = (t_{\preceq}(\gamma_n) + t_{\preceq}(\gamma_m))/2$ .

Note that  $\Gamma$  acts naturally on  $t_{\preceq}(\Gamma)$  by  $\gamma(t(\gamma_i)) = t_{\preceq}(\gamma\gamma_i)$ , and that this action extends continuously to the closure of  $t_{\preceq}(\Gamma)$ . Finally, one can extend the action to the whole real line by declaring the map  $\gamma$  to be affine on each interval of the complement of  $\overline{t_{\preceq}(\Gamma)}$ .  $\square$

We have just constructed an embedding of a countable, left-orderable group  $\Gamma$  into  $\text{Homeo}_+(\mathbb{R})$ . We call this embedding a dynamical realization of the left-ordered group  $(\Gamma, \preceq)$ . The order preserving map  $t_{\preceq}$  is called the reference map.

**Remark 1.2.** As constructed above, the dynamical realization depends not only on the left-ordering  $\preceq$ , but also on the enumeration  $(\gamma_i)_{i \geq 0}$ . Nevertheless, it is not hard to check that dynamical realizations associated to different enumerations (but the same ordering) are *topologically*

conjugate.<sup>2</sup> Thus, up to topological conjugacy, the dynamical realization depends only on the ordering  $\preceq$  of  $\Gamma$ .

An important property of dynamical realizations is that they do not admit global fixed points (i.e., no point is stabilized by the whole group). Another important property is that  $0 = t_{\preceq}(id)$  has a *free orbit* (i.e.  $\{\gamma \in \Gamma \mid \gamma(t_{\preceq}(id)) = t_{\preceq}(id)\} = \{id\}$ ). Hence  $\gamma \succ id$  if and only if  $\gamma(t_{\preceq}(id)) = \gamma(0) > 0 = t_{\preceq}(id)$ , which allows us to recover the left-ordering from its dynamical realization.

The following well-known Proposition will serve us to approximate a given left-ordering by looking at its dynamical realization. For the reader convenience, we sketch the proof below.

**Proposition 1.3.** *Let  $\Gamma$  be a left-orderable group, and let  $D : \Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$  be a (not necessarily faithful) homomorphism. Let  $x_0 \in \mathbb{R}$  and let  $\preceq_{x_0}$  be the partial-left-ordering defined by  $\gamma \succ_{x_0} id$  if and only if  $D(\gamma)(x_0) > x_0$ . Then  $\preceq_{x_0}$  can be extended to a (total) left-ordering  $\preceq$  such that  $\gamma \succ_{x_0} id$  implies  $\gamma \succ id$ .*

*Sketch of proof:* Let  $H = \{\gamma \in \Gamma \mid D(\gamma)(x_0) = x_0\}$ . Let  $\preceq'$  be any left-ordering on  $H$ . Define  $\preceq$  by

$$g \succ id \Leftrightarrow \begin{cases} D(\gamma)(x_0) > x_0 \text{ or} \\ D(\gamma)(x_0) = x_0 \text{ and } g \succ' id. \end{cases}$$

Showing that  $\preceq$  is a left-ordering on  $\Gamma$  is straightforward.  $\square$

**Definition 1.4.** Let  $\preceq$  be a left-ordering on a countable group  $\Gamma$ . Let  $D : \Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$  be an homomorphic embedding with the property that *there exists  $x \in \mathbb{R}$  such that, for  $\gamma_1$  and  $\gamma_2$  in  $\Gamma$ , we have that  $\gamma_1 \prec \gamma_2$  if and only if  $D(\gamma_1)(x) < D(\gamma_2)(x)$* . We call  $D$  a *dynamical realization-like homomorphism* for  $\preceq$ . The point  $x$  is called *reference point* for  $D$ .

**Example 1.5.** The embedding given by any dynamical realization of any countable left-ordered group  $(\Gamma, \preceq)$  is a dynamical realization-like homomorphism for  $\preceq$  with reference point  $0 = t_{\preceq}(id)$ .

**Remark 1.6.** Note that, if  $D$  is a dynamical realization-like homomorphism for  $\preceq$ , with reference point  $x$ , and if  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is any increasing homeomorphism, then the conjugated homomorphism  $D_\varphi$  defined by  $D_\varphi(g) = \varphi D(g) \varphi^{-1}$  is again a dynamical realization-like homomorphism for  $\preceq$  but with reference point  $\varphi(x)$ .

For the rest of this section,  $\Gamma$  will be a countable (not necessarily finitely generated) left-orderable group, and  $\Gamma_0$  a finite subset of  $\Gamma$  such that  $\Gamma_0 = \Gamma_0^{-1}$ . We will also denote  $\langle \Gamma_0 \rangle$  the subgroup generated by  $\Gamma_0$ . Finally, for  $w \in \langle \Gamma_0 \rangle$ , we will denote by  $|w|_{\Gamma_0}$  the word length of  $w$  with respect to  $\Gamma_0$ .

The following notion will be essential in our work.

**Definition 1.7.** Let  $B_{\Gamma_0}(n) = \{w \in \langle \Gamma_0 \rangle \mid |w|_{\Gamma_0} \leq n\}$  be the *ball* of radius  $n$  in  $\langle \Gamma_0 \rangle$ . Given  $B_{\Gamma_0}(n) \subseteq \Gamma$  and a left-ordering  $\preceq$  of  $\Gamma$ , we let

$$\lambda_{(B_{\Gamma_0}(n), \preceq)}^- = \min_{\preceq} \{w \in B_{\Gamma_0}(n)\}, \quad \lambda_{(B_{\Gamma_0}(n), \preceq)}^+ = \max_{\preceq} \{w \in B_{\Gamma_0}(n)\}.$$

Now, let  $D$  be a dynamical realization-like homomorphism for  $\preceq$ , with reference point  $x$ . Then, we will refer to the square  $[D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^-)(x), D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^+)(x)]^2 \subset \mathbb{R}^2$  as the  $(B_{\Gamma_0}(n), \preceq)$ -box.

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<sup>2</sup>Two actions  $\phi_1 : \Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$  and  $\phi_2 : \Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$  are topologically conjugate if there exists  $\varphi \in \text{Homeo}_+(\mathbb{R})$  such that  $\varphi \circ \phi_1(\gamma) = \phi_2(\gamma) \circ \varphi$  for all  $\gamma \in \Gamma$ .

**Remark 1.8.** Note that, from the left-invariance of  $\preceq$ , we have that  $|\lambda_{(B_{\Gamma_0}(n), \preceq)}^\pm|_{\Gamma_0} = n$ , and that there is  $\delta_n^+ \in \Gamma_0$  (resp.  $\delta_n^- \in \Gamma_0$ ) such that  $\delta_n^+ \lambda_{(B_{\Gamma_0}(n), \preceq)}^+ = \lambda_{(B_{\Gamma_0}(n+1), \preceq)}^+$  (resp.  $\delta_n^- \lambda_{(B_{\Gamma_0}(n), \preceq)}^- = \lambda_{(B_{\Gamma_0}(n+1), \preceq)}^-$ ).

Now let  $\preceq$  be a left-ordering on  $\Gamma$ . The next lemma shows that the  $(B_{\Gamma_0}(n), \preceq)$ -box contains the information of the  $\preceq$ -signs of the elements in  $B_{\Gamma_0}(n)$ .

**Lemma 1.9.** *Let  $D : \Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$  be a dynamical realization-like homomorphism for  $\preceq$  with reference point  $x$ . Then, for every  $w_1$  and  $w_2$  in  $B_{\Gamma_0}(n)$ , we have that  $D(w_1)(x)$  belongs to  $[D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^-)(x), D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^+)(x)]$ , and  $D(w_1)(x) > D(w_2)(x)$  if and only if  $w_1 \succ w_2$ .*

*Moreover, for any representation  $\tilde{D} : \Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$  such that, for every  $\gamma \in \Gamma_0$ , the graphs<sup>3</sup> of  $\tilde{D}(\gamma)$  coincide with the graphs of  $D(\gamma)$  inside  $[D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^-)(x), D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^+)(x)]^2$ , we have that  $D(w)(x) = \tilde{D}(w)(x)$  for all  $w \in B_{\Gamma_0}(n)$ .*

*Proof:* From Definition 1.4, it follows that for any  $w_1$  and  $w_2$  in  $\Gamma$ ,  $D(w_1)(x) > D(w_2)(x)$  if and only if  $w_1 \succ w_2$ . Now, for  $w \in B_{\Gamma_0}(n)$ , we have that  $\lambda_{(B_{\Gamma_0}(n), \preceq)}^- \preceq w \preceq \lambda_{(B_{\Gamma_0}(n), \preceq)}^+$ . In particular,  $D(w)(x) \in [D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^-)(x), D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^+)(x)]$ , which shows the first part of the lemma.

To show the second part, we note that every initial segment  $w_1$  of any reduced<sup>4</sup> word  $w \in B_{\Gamma_0}(n)$  lies again in  $B_{\Gamma_0}(n)$ . Hence, if  $w = \alpha_j \dots \alpha_1$ ,  $j \leq n$ , where  $\alpha_i \in \Gamma_0 = \Gamma_0^{-1}$ , is a reduced word, then the points  $x_1 = D(\alpha_1)(x)$ ,  $x_2 = D(\alpha_2)(x_1)$ ,  $\dots$ ,  $x_j = D(\alpha_j)(x_{j-1}) = D(w)(x)$ , they all belong to  $[D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^-)(x), D(\lambda_{(B_{\Gamma_0}(n), \preceq)}^+)(x)]$ . In particular,  $x_1 = D(\alpha_1)(x) = \tilde{D}(\alpha_1)(x), \dots, x_j = D(\alpha_j)(x_{j-1}) = \tilde{D}(\alpha_j)(x_{j-1})$ , which shows that  $D(w)(x) = \tilde{D}(w)(x)$ .  $\square$

## 2 Proof of Theorem A

### 2.1 The case where $\Gamma = G * H$ is finitely generated

Recall that the space of left-orderings of a countable group  $\Gamma$  is metrizable [13, 15, 20]. For instance, if  $\Gamma$  is finitely generated, and  $B_n$  denote the ball of radius  $n$  with respect to a finite generating set, then we can declare  $\text{dist}(\preceq_1, \preceq_2) = 1/n$ , if  $B_n$  is the largest ball on which  $\preceq_1$  and  $\preceq_2$  coincide. In particular, if  $\mathcal{LO}(\Gamma)$  contains no isolated points, then  $(\mathcal{LO}(\Gamma), \text{dist})$  becomes a compact, Hausdorff and locally disconnected metric space that has no isolated points. Hence it is homeomorphic to the Cantor set [7].

For the rest of this section,  $\Gamma$  will be the free product  $G * H$ . Both groups  $G$  and  $H$  are assumed to be finitely generated and left-orderable. The generating set of  $G$  and  $H$  will be denoted  $G_0 = \{g_1, \dots, g_k\}$  and  $H_0 = \{h_1, \dots, h_\ell\}$  respectively. We assume that  $G_0$  and  $H_0$  are closed under inversion. In particular,  $\Gamma = G * H$  is generated by  $\Gamma_0 = \{g_1, \dots, g_k, h_1, \dots, h_\ell\} = \Gamma_0^{-1}$ . Since in this case we have that  $\langle \Gamma_0 \rangle = \Gamma$ , we will denote the sets  $B_{\Gamma_0}(n)$  (see Definition 1.7) simply by  $B_n$ .

**Theorem 2.1.** *No left-ordering on  $G * H$  is isolated. In particular,  $\mathcal{LO}(G * H)$  is homeomorphic to the Cantor set.*

*Proof:* To prove Theorem 2.1, it is enough to show that, given a left-ordering  $\preceq$  and a finite subset  $F$  of  $\Gamma$ , there is a left-ordering  $\preceq'$  different from  $\preceq$  such that  $\preceq'$  coincides with  $\preceq$  over  $F$ .

<sup>3</sup>As usual, for  $f \in \text{Homeo}_+(\mathbb{R})$ , the set  $\{(x, f(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$  is called the graph of  $f$ .

<sup>4</sup>By “reduced” we mean a word of minimal length among words in  $\Gamma_0$ .

To show this, we will perform a perturbation of the dynamical realization  $D : \Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$  of  $\preceq$ . This perturbation will be made by conjugating the action of one of the factors by an order preserving homeomorphism  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , while keeping the action of the second factor untouched. As explained in Remark 1.2, we have that  $\gamma \succ \gamma'$  if and only if  $D(\gamma)(0) > D(\gamma')(0)$  for all  $\gamma, \gamma'$  in  $\Gamma$ .

Since  $\preceq$  is fixed, to avoid heavy notation, we will denote the elements  $\lambda_{(B_n, \preceq)}^\pm$  simply by  $\lambda_n^\pm$ . We now let  $n \in \mathbb{N}$  be such that  $F \subseteq B_n$ .

Now, consider  $\lambda_{n+1}^+$ , and let  $g \in G_0$  and  $h \in H_0$  be such that  $g\lambda_{n+1}^+ \succ \lambda_{n+1}^+$  and  $h\lambda_{n+1}^+ \succ \lambda_{n+1}^+$ . Since we are not making any different assumption on  $G$  and  $H$ , we can assume that  $g\lambda_{n+1}^+ \succ h\lambda_{n+1}^+$  (otherwise we change the names...).

We also let  $x_0, x_1, y_0, y_1$  in  $\mathbb{R}$  be such that

$$D(\lambda_{n+1}^+)(0) < x_0 < x_1 < D(h\lambda_{n+1}^+)(0) < D(g\lambda_{n+1}^+)(0) < y_1 < y_0.$$

We let  $\varphi \in \text{Homeo}_+(\mathbb{R})$  be such that  $\text{supp}(\varphi) = \{x \in \mathbb{R} \mid \varphi(x) \neq x\} = (x_0, y_0)$  and  $\varphi(x_1) > y_1$ . This implies that

$$\varphi \circ D(h\lambda_{n+1}^+) \circ \varphi^{-1}(0) > D(g\lambda_{n+1}^+)(0), \quad (1)$$

where  $\circ$  is the composition operation. Moreover, for any  $\bar{h} \in H_0$  and any  $x \in [D(\lambda_n^-)(0), D(\lambda_n^+)(0)]$ , we have that  $D(\bar{h})(x) \leq D(\lambda_{n+1}^+)(0) < x_0$ . Thus we conclude,

$$\varphi \circ D(\bar{h}) \circ \varphi^{-1}(x) = D(\bar{h})(x), \text{ for all } x \leq D(\lambda_n^+)(0) \text{ and all } \bar{h} \in H_0. \quad (2)$$

Now, let  $D_\varphi : \Gamma \rightarrow \text{Homeo}_+(\mathbb{R})$  be defined by  $D_\varphi(\bar{g}) = D(\bar{g})$  for all  $\bar{g} \in G$ , and  $D_\varphi(\bar{h}) = \varphi \circ D(\bar{h}) \circ \varphi^{-1}$  for all  $\bar{h} \in H$ . Since  $\Gamma$  is the free product of  $G$  and  $H$ , we have that  $D_\varphi$  is an homomorphism (not necessarily injective). Now, from the definition of  $D_\varphi$  and equation (2), we have that

$$D(\gamma)(x) = D_\varphi(\gamma)(x) \text{ for any } \gamma \in \Gamma_0 \text{ and any } x \leq D(\lambda_n^+)(0). \quad (3)$$

In particular, for each  $\gamma \in \Gamma_0$ , the graphs of  $D(\gamma)$  and  $D_\varphi(\gamma)$  coincide inside the square  $[D(\lambda_n^-)(0), D(\lambda_n^+)(0)]^2$ . Hence, from Lemma 1.9, we conclude that

$$\text{for all } \gamma \in B_n, \quad D(\gamma)(0) = D_\varphi(\gamma)(0). \quad (4)$$

Now, from Lemma 1.3, we have that there is a left-ordering  $\preceq'$  on  $\Gamma$  such that  $D_\varphi(\gamma)(0) > 0$  implies  $\gamma \succ' id$ . Then, equation (4) implies that  $\preceq$  and  $\preceq'$  coincide on  $B_n$ , hence, on  $F$ .

However, if we let  $\delta_n \in \Gamma_0$  be such that  $\delta_n \lambda_n^+ = \lambda_{n+1}^+$  (see Remark 1.8), we have that  $D(g\lambda_{n+1}^+)(0) = D(g) \circ D(\delta_n) \circ D(\lambda_n^+)(0)$ . Hence, from the definition of  $D_\varphi$  and equations (3) and (4), we conclude that  $D(g\lambda_{n+1}^+)(0) = D_\varphi(g\lambda_{n+1}^+)(0)$ . Moreover, from the definition of  $\varphi$ , we have that  $\varphi \circ D(h\lambda_{n+1}^+) \circ \varphi^{-1}(0) = D_\varphi(h) \circ \varphi \circ D(\lambda_{n+1}^+) \circ \varphi^{-1}(0) = D_\varphi(h) \circ D(\lambda_{n+1}^+)(0) = D_\varphi(h) \circ D(\delta_n) \circ D(\lambda_n^+)(0)$ . Therefore, equations (3) and (4) imply that  $\varphi \circ D(h\lambda_{n+1}^+) \circ \varphi^{-1}(0) = D_\varphi(h\lambda_{n+1}^+)(0)$ . Hence, equation (1) reads

$$D_\varphi(h\lambda_{n+1}^+)(0) > D_\varphi(g\lambda_{n+1}^+)(0),$$

which implies that  $h\lambda_{n+1}^+ \succ' g\lambda_{n+1}^+$ . In particular, we have that  $\preceq'$  is different from  $\preceq$  because we had assumed that  $h\lambda_{n+1}^+ \prec g\lambda_{n+1}^+$ . This finishes the proof of Theorem 2.1.  $\square$

## 2.2 The general case

There is a well-known criterion from Conrad-Ohnishi [3, 17] stating that a group  $\Gamma$  is left-orderable if and only if for every finite family  $f_1, \dots, f_k$ , all of them different from the identity, there exist  $\eta_i \in \{-1, 1\}$ ,  $i = 1, \dots, k$ , such that the identity is not contained in the smallest semigroup containing  $\{f_1^{\eta_1}, \dots, f_k^{\eta_k}\}$ . We will denote this semigroup by  $\langle f_1^{\eta_1}, \dots, f_k^{\eta_k} \rangle^+$ .

In [15, Proposition 1.4], Navas shows that this criterion (and the analogous one for bi-orderings [17] and Conradian orderings) is closely related to the compactness of  $\mathcal{LO}(\Gamma)$ . Below, we present an extension of this criterion that will permit us to deduce Theorem A from our proof of Theorem 2.1. This extension may be found in [10, Lemma 3.1.1]. However, for completeness, we give a proof of it.

Let  $\gamma_1, \dots, \gamma_n$  be a finite family of non-trivial elements in a group  $\Gamma$ . We say that  $\gamma_1, \dots, \gamma_n$  has property (E) if and only if

(E): *for every finite family  $f_1, \dots, f_k$ , of elements different from the identity, there exists  $\eta_i \in \{-1, 1\}$ ,  $i = 1, \dots, k$ , such that  $id \notin \langle \gamma_1, \dots, \gamma_n, f_1^{\eta_1}, \dots, f_k^{\eta_k} \rangle^+$ .*

We say that such a choice of exponents  $\eta_i$  is *compatible*.

**Lemma 2.2.** *Let  $\gamma_1, \dots, \gamma_n$  be non trivial elements in a left-orderable group  $\Gamma$ . Then  $\Gamma$  admits a left-ordering  $\preceq$  such that  $\gamma_i \succ id$  for all  $i = 1, \dots, n$ , if and only if  $\gamma_1, \dots, \gamma_n$  has property (E).*

*Proof:* The necessity of property (E) is obvious.

To see the sufficiency we will use the compactness of  $\mathcal{PLO}(\Gamma)$ . For each finite family  $f_1, \dots, f_k$  of non-trivial elements in  $\Gamma$ , and each compatible choice of  $\eta_i$ , we let  $\chi(f_1, \dots, f_k; \eta_1, \dots, \eta_k)$  be the (closed) set of all partial-left-orderings such that each  $\gamma_j$ ,  $j = 1, \dots, n$ , and each  $f_\ell^{\eta_\ell}$ ,  $\ell = 1, \dots, k$ , is positive. By hypothesis, this set is non-empty.

Now, let  $\chi(f_1, \dots, f_k)$  be the (finite) union of all the sets of the form  $\chi(f_1, \dots, f_k; \eta_1, \dots, \eta_k)$ , where the choice of the exponents  $\eta_i$  is compatible. Note that if  $\{\chi_i = \chi(f_{i,1}, \dots, f_{i,k}); i = 1, \dots, n\}$  is a finite family of subsets of this form, then, the intersection  $\chi_1 \cap \dots \cap \chi_n$  contains (the non-empty)  $\chi(f_{1,1}, \dots, f_{1,k}, \dots, f_{n,1}, \dots, f_{n,k})$ . Since  $\mathcal{PLO}(\Gamma)$  is compact, a direct application of the finite intersection property shows that  $\chi$ , the intersection of all the sets of the form  $\chi(f_1, \dots, f_k)$ , is non-empty. It is quite clear that any partial-left-ordering  $\preceq \in \chi$  is a total ordering of  $\Gamma$ . Hence, any left-ordering on  $\chi$  is a left-ordering in which each  $\gamma_i$ ,  $i = 1, \dots, n$ , is positive  $\square$

We now pass to the Proof of Theorem A.

Let  $\preceq$  be a left-ordering on  $G * H$ , and let  $F$  be a finite subset of  $\preceq$ -positive elements in  $G * H$  on which we want to approximate  $\preceq$ . Let  $G_0 \subset G$  and  $H_0 \subset H$  be two finite non-empty sets such that  $G_0 = G_0^{-1}$ ,  $H_0 = H_0^{-1}$  and such that  $F \subset \langle G_0 \rangle * \langle H_0 \rangle$ . Let  $\Gamma_0 = G_0 \cup H_0 = \Gamma_0^{-1}$  and  $\Gamma = \langle \Gamma_0 \rangle$ .

Let  $n \in \mathbb{N}$  be such that  $F \subset B_{\Gamma_0}(n)$  and let  $g \in G_0$  and  $h \in H_0$  be such that  $\lambda_{(B_{\Gamma_0}(n), \preceq)}^+ \prec h\lambda_{(B_{\Gamma_0}(n), \preceq)}^+$  and  $\lambda_{(B_{\Gamma_0}(n), \preceq)}^+ \prec g\lambda_{(B_{\Gamma_0}(n), \preceq)}^+$  (see Definition 1.7). As in the proof of Theorem 2.1, we may also assume that  $h\lambda_{(B_{\Gamma_0}(n), \preceq)}^+ \prec g\lambda_{(B_{\Gamma_0}(n), \preceq)}^+$  (otherwise, we change the names of  $G$  and  $H$ ). Finally, let  $\gamma_* = (h\lambda_{(B_{\Gamma_0}(n), \preceq)}^+)^{-1}g\lambda_{(B_{\Gamma_0}(n), \preceq)}^+$ . Note that  $id \prec \gamma_*$ .

Theorem A follows directly from

**Claim A:** The set  $F \cup \{\gamma_*^{-1}\}$  has property (E).

In its turn, Claim A follows directly from

**Lemma 2.3.** *With the notations above, for any finitely generated subgroup  $\hat{\Gamma}$  of  $G * H$  such that  $\Gamma \subset \hat{\Gamma}$ , there exists a left-ordering  $\preceq^*$  on  $\hat{\Gamma}$  such that any element in  $F \cup \{\gamma_*^{-1}\}$  is  $\preceq^*$ -positive.*

*Proof:* The proof follows the same lines as the proof of Theorem 2.1. Fix  $\hat{\Gamma}$  a finitely generated subgroup of  $G * H$  containing  $\Gamma$ . We let  $\hat{\Gamma}_0$  be the generating set of  $\hat{\Gamma}$ . By eventually enlarging  $\hat{\Gamma}$ , we shall assume that  $\hat{\Gamma}_0 = \hat{G}_0 \cup \hat{H}_0$ , where  $\hat{G}_0 \subset G$  and  $\hat{H}_0 \subset H$ , both non-empty sets. In this way we have that  $\hat{\Gamma}_0 = \langle \hat{G}_0 \rangle * \langle \hat{H}_0 \rangle$ .

To avoid heavy notation, for any  $k \in \mathbb{N}$ , we let  $\lambda_k^+ = \lambda_{(B_{\Gamma_0}(k), \preceq)}^+$  and  $\lambda_k^- = \lambda_{(B_{\Gamma_0}(k), \preceq)}^-$ .

We let  $D : \hat{\Gamma} \rightarrow \text{Homeo}_+(\mathbb{R})$  be the dynamical realization of the restriction of  $\preceq$  to  $\hat{\Gamma}$ , that is, for any  $\gamma \in \hat{\Gamma}$ ,  $\gamma \succ id$  if and only if  $D(\gamma)(0) > 0$ .

We let  $x_0, x_1, y_0, y_1$  in  $\mathbb{R}$  be such that

$$D(\lambda_{n+1}^+)(0) < x_0 < x_1 < D(h\lambda_{n+1}^+)(0) < D(g\lambda_{n+1}^+)(0) < y_1 < y_0.$$

We let  $\varphi \in \text{Homeo}_+(\mathbb{R})$  be such that  $\text{supp}(\varphi) = \{x \in \mathbb{R} \mid \varphi(x) \neq x\} = (x_0, y_0)$  and that  $\varphi(x_1) > y_1$ . This implies that

$$\varphi \circ D(h\lambda_{n+1}^+) \circ \varphi^{-1}(0) > D(g\lambda_{n+1}^+)(0). \quad (5)$$

Moreover, for any  $\bar{h} \in H_0$ , and any  $x \in [D(\lambda_n^-)(0), D(\lambda_n^+)(0)]$ , we have that  $D(\bar{h})(x) \leq D(\lambda_{n+1}^+)(0) < x_0$ . Thus we conclude,

$$\varphi \circ D(\bar{h}) \circ \varphi^{-1}(x) = D(\bar{h})(x), \text{ for all } x \leq D(\lambda_n^+)(0), \text{ and all } \bar{h} \in H_0. \quad (6)$$

Now, let  $D_\varphi : \hat{\Gamma} \rightarrow \text{Homeo}_+(\mathbb{R})$  be defined by  $D_\varphi(\bar{g}) = D(\bar{g})$  for all  $\bar{g} \in \langle \hat{G}_0 \rangle$ , and  $D_\varphi(\bar{h}) = \varphi \circ D(\bar{h}) \circ \varphi^{-1}$  for all  $\bar{h} \in \langle \hat{H}_0 \rangle$ . Since  $\hat{\Gamma}$  is the free product of  $\langle \hat{G}_0 \rangle$  and  $\langle \hat{H}_0 \rangle$ , we have that  $D_\varphi$  is an homomorphism (not necessarily injective). Now, from the definition of  $D_\varphi$  and equation (6), we have that

$$D(\gamma)(x) = D_\varphi(\gamma)(x), \text{ for all } \gamma \in \Gamma_0 \text{ and any } x \leq D(\lambda_{n+1}^+)(0).$$

In particular, for each  $\gamma \in \Gamma_0$ , the graphs of  $D(\gamma)$  and  $D_\varphi(\gamma)$  coincide inside the square  $[D(\lambda_n^-)(0), D(\lambda_n^+)(0)]^2$ . Hence, from Lemma 1.9, we have that

$$\text{for all } \gamma \in B_{\Gamma_0}(n), \quad D(\gamma)(0) = D_\varphi(\gamma)(0). \quad (7)$$

Now, from Lemma 1.3, there is a left-ordering  $\preceq^*$  on  $\hat{\Gamma}$  such that  $D_\varphi(\gamma)(0) > 0$  implies  $\gamma \succ^* id$ . Then, equation (7) implies that  $\preceq$  and  $\preceq^*$  coincide on  $B_{\Gamma_0}(n)$ . In particular, any element in  $F$  is  $\preceq^*$ -positive.

However, arguing as in the end of the proof of Theorem 2.1, it can be shown that equation (5) is the same as

$$D_\varphi(h\lambda_{n+1}^+)(0) > D_\varphi(g\lambda_{n+1}^+)(0),$$

which shows that  $id \prec^* \gamma_*^{-1}$ . □

This finishes the proof of Theorem A.



### 2.3 An example

We have proved that no left-ordering on a free product of groups is isolated. In particular no positive cone of a left-ordering on a free product is finitely generated as a semigroup [15, Proposition 1.8]. In this section, we show that there exist a group with an isolated left-ordering whose positive cone is not finitely generated as a semigroup. This seems to be the first example of a group with this property.

**Proposition 2.4.** *The group  $\Gamma = \langle a, b \mid bab^{-1} = a^{-2} \rangle$  is a finitely generated group with an isolated left-ordering whose positive cone is not finitely generated as a semigroup.*

*Proof:* The group  $\Gamma$  is a group fitting in the classification of groups having only finitely many left-orderings [10, Theorem 5.2.1]. However, we shall provide a direct argument showing that it contains an isolated left-ordering.

Let  $\Gamma_1$  be the subgroup generated by  $\{b^j ab^{-j} \mid j \in \mathbb{Z}\}$ , and let  $m, n$  in  $\mathbb{Z}$ . Note that both  $b^n ab^{-n}$  and  $b^m ab^{-m}$  belong to  $\langle b^k ab^{-k} \rangle$ , where  $k = \min\{0, n, m\}$ . In particular,  $\Gamma_1$  is an Abelian group which is isomorphic to a non cyclic subgroup of the rational numbers. Furthermore,  $\Gamma_1$  is normal in  $\Gamma$  and the quotient  $\Gamma/\Gamma_1 = \langle b\Gamma_1 \rangle$  is isomorphic to  $\mathbb{Z}$ .

We let  $\preceq_*$  be a left-ordering of  $\Gamma_1$  such that  $a \succ_* id$ , and  $\preceq^*$  be a left-ordering on  $\Gamma/\Gamma_1$  such that  $b\Gamma_1 \succ^* \Gamma_1$ . In this way we can left-order  $\Gamma$  by declaring

$$g \succ id \Leftrightarrow \begin{cases} g\Gamma_1 \neq \Gamma_1 \text{ and } g\Gamma_1 \succ^* \Gamma_1, \text{ or} \\ g \in \Gamma_1 \text{ and } g \succ_* id. \end{cases}$$

We claim that  $\preceq$  is an isolated left-ordering. Indeed, let  $\preceq'$  be a left-ordering such that  $b \succ' id$  and such that  $a \succ' id$ . In particular, since  $\Gamma_1$  is isomorphic to a subgroup of the rational numbers, we have that  $\preceq'$  coincide with  $\preceq$  on  $\Gamma_1$ . Now let  $g \in \Gamma$  be such that  $g \notin \Gamma_1$ . Let  $n \in \mathbb{Z} \setminus \{0\}$  be such that  $b^n \Gamma_1 = g\Gamma_1$ , that is,  $g = b^n g_1$  for some  $g_1 \in \Gamma_1$ . Suppose first that  $n \geq 1$ . In this case we have that  $g = b^n g_1 = b^{n-1} g_1^{-2} b$ , which shows that we can write  $g$  as a product of  $\preceq'$ -positive elements. In particular  $g \succ' id$ . In the case that  $n \leq -1$ , the preceding argument shows that  $g^{-1}$  is  $\preceq'$ -positive. Hence, we have that  $\preceq'$  coincide with  $\preceq$  on  $\Gamma$ , showing that  $\preceq$  is an isolated left-ordering.

Now, suppose by way of a contradiction that  $\preceq$  has a positive cone which is finitely generated as a semigroup. That is,  $P_{\preceq} = \{\gamma \in \Gamma \mid \gamma \succ id\} = \langle S \rangle^+$ , where  $S = \{\gamma_1, \dots, \gamma_n\}$ . By eventually re-labeling  $S$ , we may assume that  $S = \{\gamma_1, \dots, \gamma_j, \dots, \gamma_n\}$ , where  $\gamma_i \Gamma_1 \succ^* \Gamma_1$ , for  $1 \leq i \leq j$ , and  $\gamma_i \in \Gamma_1$ , for  $i > j$ . For  $1 \leq i \leq j$  we let  $\gamma_i = b^{n_i} g_i$ , where  $n_i \geq 1$  and  $g_i \in \Gamma_1$ .

Now let  $w = \gamma_{m_1} \dots \gamma_{m_k}$  be an element in  $\langle S \rangle^+$ . Since  $\preceq^*$  is a left-ordering, we have that  $w\Gamma_1 \succeq^* \Gamma_1$ . This implies that any  $\preceq$ -positive  $g \in \Gamma_1$  may be written as a product of  $\gamma_{j+1}, \dots, \gamma_n$ . However, this is impossible since  $\Gamma_1$  is a non-cyclic subgroup of the rational numbers. This settles the desired contradiction.  $\square$

## 3 Constructing a dense orbit in the space of left-orderings of the free group

We now proceed to the the construction of a left-ordering on the free group of countable rank greater than one whose orbit is dense under the natural conjugation action. The rough idea is the following. Since the space of left-orderings of a countable group is a compact metric space (see for instance [13, 15, 20] or the beginning of §2.1), it contains a dense countable subset. Now, we can consider the *dynamical realization* (see §1.1) of each of these left-orderings, and cut large pieces from each one of them (see for instance Definition 1.7). Since we are working with a free group,

we can glue these pieces of dynamical realizations together in a sole action of our group on the real line. Moreover, if the gluing is made with a little bit of care, then we can ensure very nice conjugacy properties from which we can deduce Theorem B.

First, we define an enumeration of the set of *balls* on a countable free group. Let  $S_\omega^+ = \{a, b, \alpha_1, \alpha_2, \dots\}$  be a free generating set of the free group of countable infinite rank  $F_\omega$ . For  $m \in \mathbb{N} = \{1, 2, \dots\}$ , we let  $S_m^+ = \{a, b, \alpha_1, \dots, \alpha_{m-2}\}$  if  $m \geq 2$ , and  $S_1^+ = \{a\}$ . For  $n \in \mathbb{N} \cup \{\omega\}$ , we let  $S_n = S_n^+ \cup (S_n^+)^{-1}$ . Note that we have the inclusion  $S_n \subset S_\omega$ , and that  $F_n = \langle S_n \rangle$ . Using the notations of Definition 1.7, we let

$$\mathcal{B}(F_n) = \begin{cases} \{B_{S_n}(m) \mid m \in \mathbb{N}\} & \text{if } n \neq \omega, \\ \{B_{S_m}(m) \mid m \in \mathbb{N}\} & \text{if } n = \omega. \end{cases}$$

We call  $\mathcal{B}(F_n)$  the set of *balls* in  $F_n$ . We define  $\phi_n : \mathbb{N} \rightarrow \mathcal{B}(F_n)$  by  $\phi_n(m) = B_{S_n}(m)$  if  $n \neq \omega$  and  $\phi_\omega(m) = B_{S_{m+1}}(m+1)$ . Note that,  $\cup_{m \in \mathbb{N}} \phi_n(m) = F_n$  and that, for any  $B \in \mathcal{B}(F_n)$ ,  $n \neq 1$ , we have that  $a$  and  $b$  belong to  $B$ . Note also that  $S_\omega \cap \phi_n(m) = S_k$ , where  $k = \min\{n, m+1\}$  (assuming that  $\omega$  is bigger than any integer).

Fix once and for all  $n \in \mathbb{N} \cup \{\omega\}$ ,  $n \neq 1$ . Let  $\phi = \phi_n$ ,  $\mathcal{B} = \mathcal{B}(F_n)$ , and  $\mathcal{D} = \{\preceq_1, \preceq_2, \dots\}$  be a countable dense subset of  $\mathcal{LO}(F_n)$ . Let  $\eta : \mathbb{Z} \rightarrow \mathcal{B} \times \mathcal{D}$  be a surjection, with  $\eta(k) = (\phi(n_k), \preceq_{m_k})$ .

By Remak 1.6 we have that there exists  $D_{\eta(k)} : F_n \rightarrow \text{Homeo}_+(\mathbb{R})$ , a dynamical realization-like homomorphism for  $\preceq_{m_k}$ , such that:

- (i) The reference point for  $D_{\eta(k)}$  is  $k$ .
- (ii) The  $\eta(k)$ -box coincides with the square  $[k - 1/3, k + 1/3]^2$ .

Theorem B is a direct consequence of the following

**Proposition 3.1.** *There is an homomorphism  $D : F_n \rightarrow \text{Homeo}_+(\mathbb{R})$  such that, for each  $k \in \mathbb{Z}$ , inside  $[k - 1/3, k + 1/3]^2$ , the graphs of  $D(g)$  coincide with the graphs of  $D_{\eta(k)}(g)$  for any  $g \in S_n \cap \phi(n_k)$ . In this action, all the integers lie in the same orbit.*

*Proof of Theorem B from Proposition 3.1:* Let  $(x_0, x_1, \dots)$  be a dense sequence in  $\mathbb{R}$  such that  $x_0 = 0$  (note that 0 may not have a free orbit), and let  $D$  be the homomorphism given by Proposition 3.1. Note that  $D$  is an embedding, since, from Lemma 1.9, we have that any non-trivial  $w \in \phi(n_k)$  acts nontrivially at the point  $k \in \mathbb{R}$ . Hence, we may let  $\preceq$  be the induced left-ordering on  $F_n$  from the action  $D$  and the reference points  $(x_0, x_1, x_2, \dots)$ . In particular, for  $h \in F_n$ , we have that  $D(h)(0) > 0 \Rightarrow h \succ id$ . We claim that  $\preceq$  has a dense orbit under the natural action of  $F_n$  on  $\mathcal{LO}(F_n)$ .

Clearly, to prove our claim it is enough to prove that the orbit of  $\preceq$  accumulates at every  $\preceq_m \in \mathcal{D}$ . That is, given  $\preceq_m$  and any finite set  $\{h_1, h_2, \dots, h_N\}$  such that  $id \prec_m h_j$ , for  $1 \leq j \leq N$ , we need to find  $w \in F_n$  such that  $h_j \succ_w id$  for every  $1 \leq j \leq N$ , where, as defined in the Introduction,  $h \succ_w id$  if and only if  $whw^{-1} \succ id$ .

Let  $j \in \mathbb{N}$  be such that  $h_1, \dots, h_N$  belongs to  $\phi(j)$ . Let  $k$  be such that  $\eta(k) = (\phi(j), \preceq_m)$ . By Proposition 3.1, there is  $w_k \in F_n$  such that  $D(w_k)(0) = k$ . Also by Proposition 3.1, inside  $[k - 1/3, k + 1/3]^2$ , for every  $g \in S_\omega \cap \phi(j)$  we have that the graphs of  $D(g)$  are the same as those of  $D_{\eta(k)}(g)$ . Then, Lemma 1.9 implies that for each  $h_j$ ,  $1 \leq j \leq N$ , we have that  $h_j \succ_m id$  if and only if  $D(h_j)(k) > k$ . But this is the same as saying that  $D(h_j)(D(w_k)(0)) > D(w_k)(0)$ , which implies that  $D(w_k^{-1}) \circ D(h_j) \circ D(w_k)(0) > 0$ . Therefore, by definition of  $\preceq$ , we have that  $w_k^{-1} h_j w_k \succ id$  for every  $1 \leq j \leq N$ . Now, by definition of the action of  $F_n$  on  $\mathcal{LO}(F_n)$ , this implies that  $\preceq_{w_k^{-1}}$  is a left-ordering such that  $h_j \succ_{w_k^{-1}} id$ . This finishes the proof of Theorem B.  $\square$

To prove Proposition 3.1 we first consider  $g \in S_n$ , and let  $K = \{k \in \mathbb{Z} \mid g \in \phi(n_k)\}$ . Now if  $k_0$  and  $k_1$  are elements of  $K$  such that  $k_0 < k_1$  and such that there is no other element of  $K$  in between, then we can linearly interpolate the portion of the graph of  $D_{\eta(k_0)}(g)$  inside  $[k_0 - 1/3, k_0 + 1/3]^2$  until the portion of the graph of  $D_{\eta(k_1)}(g)$  inside  $[k_1 - 1/3, k_1 + 1/3]^2$ . Repeating this argument, we get a function  $\hat{g} \in \text{Homeo}_+(\mathbb{R})$  that coincides with  $D_{\eta(k)}(g)$  for all  $k \in K$ . In this way we have proved

**Lemma 3.2.** *Let  $g \in S_n$ . For each  $k \in \mathbb{Z}$  we let  $n_k$  and  $m_k$  in  $\mathbb{N}$  be such that  $\eta(k) = (\phi(n_k), \preceq_{m_k})$ . Then, there exist  $\hat{g} \in \text{Homeo}_+(\mathbb{R})$  such that for every  $k \in \mathbb{Z}$  such that  $g \in \phi(n_k)$ , the graph of  $\hat{g}$  inside  $[k - 1/3, k + 1/3]^2$  coincide with the graphs of  $D_{\eta(k)}(g)$ .*

**Lemma 3.3.** *For each  $k \in \mathbb{Z}$ , we can modify the homeomorphisms  $\hat{a}$  and  $\hat{b}$  (given by Lemma 3.2) inside  $[k - 1/3, k + 1 + 1/3]^2$  but outside  $[k - 1/3, k + 1/3]^2 \cup [k + 1 - 1/3, k + 1 + 1/3]^2$  (see Figure 4.1) in such a way that the modified homeomorphisms, which we still denote  $\hat{a}$  and  $\hat{b}$ , have the following property*

(P) : *there is a reduced word  $w$  in the free group generated by  $\{\hat{a}, \hat{b}\}$  such that  $w(k + 1/3) = k + 1 - 1/3$ . Moreover, the iterates of  $k + 1/3$  along the initial segments of  $w$  remain inside  $[k - 1/3, k + 1 + 1/3]$ .*

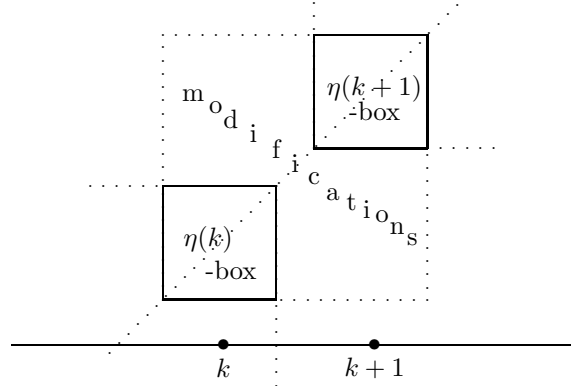


Figure 4.1

*Proof:* For  $h \in \{\hat{a}^{\pm 1}, \hat{b}^{\pm 1}\}$ , define  $l_h = \sup\{x \in [k - 1/3, k + 1/3] \mid h(x) \leq k + 1/3\}$  and  $r_h = \inf\{x \in [k + 1 - 1/3, k + 1 + 1/3] \mid h(x) \geq k + 1 - 1/3\}$ . Let  $x_0 \in ]k + 1/3, k + 1 - 1/3[$ . To modify  $\hat{a}$  and  $\hat{b}$ , we proceed as follows:

**Case 1:** There is  $h \in \{\hat{a}^{\pm 1}, \hat{b}^{\pm 1}\}$  such that  $l_h < k + 1/3$  and  $r_h = k + 1 - 1/3$ .

In this case, we (re)define  $h$  linearly from  $(l_h, h(l_h)) = (l_h, k + 1/3)$  to  $(k + 1/3, x_0)$ , then linearly from  $(k + 1/3, x_0)$  to  $(x_0, k + 1 - 1/3)$ , and then linearly from  $(x_0, k + 1 - 1/3)$  to  $(k + 1 - 1/3, h(k + 1 - 1/3)) = (r_h, h(r_h))$ ; see Figure 4.2 (a). The other generator, say  $f$ , may be extended linearly from  $(l_f, f(l_f))$  to  $(r_f, f(r_f))$ .

Note that in this case we have  $h(k + 1/3) = x_0$  and  $h(x_0) = k + 1 - 1/3$ . This shows that (P) holds for  $w = h^2$ .

We note that, for  $h \in \{\hat{a}^{\pm 1}, \hat{b}^{\pm 1}\}$ , we have that  $l_h = k + 1/3 \Leftrightarrow l_{h^{-1}} < k + 1/3$  and  $r_h = k + 1 - 1/3 \Leftrightarrow r_{h^{-1}} > k + 1 - 1/3$ . Therefore, if there is no  $h$  as in Case 1, then we are in

**Case 2:** There are  $f, h \in \{\hat{a}^{\pm 1}, \hat{b}^{\pm 1}\}$  such that  $l_h < k + 1/3$ ,  $r_h > k + 1 - 1/3$ ,  $l_f < k + 1/3$  and  $r_f > k + 1 - 1/3$ .

In this case we define  $h$  linearly from  $(l_h, h(l_h))$  to  $(k + 1/3, x_0)$ , and then linearly from  $(k + 1/3, x_0)$  to  $(r_h, h(r_h))$ . For  $f$ , we define it linearly from  $(l_f, f(l_f))$  to  $(k + 1 - 1/3, x_0)$ , and then linearly from  $(k + 1 - 1/3, x_0)$  to  $(r_f, f(r_f))$ ; see Figure 4.2 (b).

Note that  $h(k + 1/3) = x_0 = f(k + 1 - 1/3)$ . This shows that  $(P)$  holds for  $w = f^{-1}h$ .  $\square$

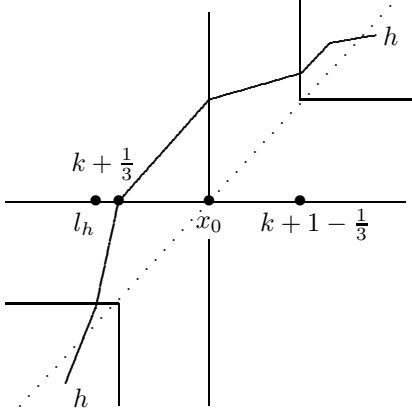


Figure 4.2 (a)

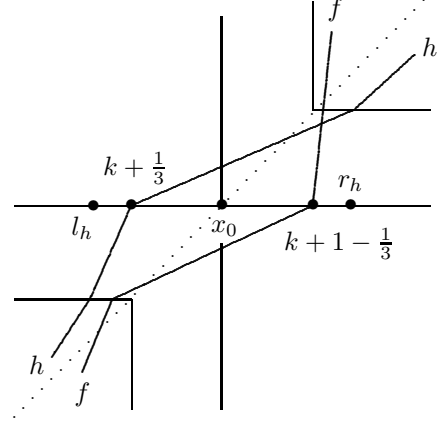


Figure 4.2 (b)

*Proof of Proposition 3.1:* For each  $g \in S_n^+$ , we let  $\hat{g}$  be as in Lemma 3.2. Hence, inside  $[k - 1/3, k + 1/3]^2$ , the graphs of  $\hat{g}$  coincide with the graphs of  $D_{\eta(k)}(g)$  for any  $g \in S_n \cap \phi(n_k)$ , where  $\eta(k) = (\phi(n_k), \preceq_{m_k})$ . Now, for each  $k \in \mathbb{Z}$  we apply inductively Lemma 3.3 to modify  $\hat{a}$  and  $\hat{b}$ . This modified homeomorphisms will be still denoted  $\hat{a}$  and  $\hat{b}$ . Note that Lemma 3.3 implies that the modifications are made in such a way that they do not overlap one with each other and that, for each  $k \in \mathbb{Z}$ , the graphs of  $\hat{a}$  and  $\hat{b}$  coincides with the graphs of  $D_{\eta(k)}(a)$  and  $D_{\eta(k)}(b)$  inside  $[k - 1/3, k + 1/3]^2$ . Therefore, if we define  $\hat{D} : F_n \rightarrow \text{Homeo}_+(\mathbb{R})$  by  $\hat{D}(g) = \hat{g}$  for every  $g \in S_n$ , we have that, inside  $[k - 1/3, k + 1/3]^2$ , the graphs of  $\hat{D}(g)$  coincide with the graphs of  $D_{\eta(k)}(g)$  for any  $g \in S_n \cap \phi(n_k)$ .

Finally, for each  $k \in \mathbb{Z}$ , Lemma 1.9 implies that in the action given by  $\hat{D}$ , the points  $k, k + 1/3$  and  $k - 1/3$  are in the same orbit. Hence, from Lemma 3.3, we have that in this action all the integers are in the same orbit. This finishes the proof of Proposition 3.1.  $\square$

## References

- [1] A. CLAY. Free lattice ordered groups and the topology on the space of left orderings of a group. To appear in *Monatshefte für Mathematik* (available online).
- [2] P. CONRAD. Free lattice-ordered groups. *J. Algebra* **16** (1970), 191-203.
- [3] P. CONRAD. Right-ordered groups. *Mich. Math. Journal* **6** (1959), 267-275.
- [4] P. DEHORNOY, I. DYNNIKOV, D. ROLFSEN & B. WIEST. *Ordering Braids*. Math. Surveys and Monographs **148**, A.M.S. (2008).
- [5] T.V. DUBROVINA & N.I. DUBROVIN. On braid groups. *Mat. Sb.* **192** (2001), 693-703.
- [6] É. GHYS. Groups acting on the circle. *L'Enseignement Mathématique* **47** (2001), 239-407.
- [7] J.G. HOCKING & G.S. YOUNG. *Topology*. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London (1961).
- [8] T. ITO. Dehornoy-like left orderings and isolated left orderings, *Preprint*. arXiv 1102.4669v1.

- [9] V. KOPYTOV. Free lattice-ordered groups. *Sibirsk Mat. Zh.* **24**(1) (1983), 120-124.
- [10] V. KOPYTOV & N. MEDVEDEV. *Right ordered groups*. Siberian School of Algebra and Logic, Plenum Publ. Corp., New York (1996).
- [11] P. LINNELL. The space of left orders of a group is either finite or uncountable. *London Math. Soc.* **43** (2011), 200-202.
- [12] S.H. MCCLEARY. Free lattice-ordered group represented as  $o$ -2 transitive  $l$ -permutation groups. *Trans. Amer. Math. Soc.*, **290**(1) (1985), 69-79.
- [13] D. MORRIS-WITTE. Amenable groups that act on the line. *Algebr. Geom. Topol.* **6** (2006), 2509-2518.
- [14] A. NAVAS. A remarkable family of left-ordered groups: central extensions of Hecke groups. *J. Algebra* **328** (2011), 31-42.
- [15] A. NAVAS. On the dynamics of (left) orderable groups. *Ann. Inst. Fourier (Grenoble)* **60** (2010), 1685-1740.
- [16] A. NAVAS & B. WIEST. Nielsen-Thurston orderings and the space of braid orderings. To appear in *Bull. of Lon. Math. Soc.* (available online).
- [17] M. OHNISHI. Linear order on a group. *Osaka Math. J.* **2** (1959), 17-18.
- [18] C. RIVAS. On groups with finitely many Conradian orderings. To appear in *Comm. in Algebra*.
- [19] C. RIVAS. Orderable groups. Ph.D. thesis, Univ. de Chile (2010).
- [20] A. SIKORA. Topology on the spaces of orderings of groups. *Bull. London Math. Soc.* **36** (2004), 519-526.

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